JOURNAL OF COMBINATORIAL THEORY, Series A 55, 33–48 (1990)

Solution of a Delta-System Decomposition Problem

ZBIGNIEW LONC

Institute of Mathematics, Warsaw University of Technology, 00–661 Warsaw, Poland

Communicated by the Managing Editors

Received October 3, 1988

A decomposition of a hypergraph H into hypergraphs $H_1, ..., H_q$ is a partition of the edge set $\mathscr{E}$ of H into subsets $\mathscr{E}_1, ..., \mathscr{E}_q$ such that H_i is generated by $\mathscr{E}_i$, for $i = 1, ..., q$. An h-uniform hypergraph generated by edges $E_1, ..., E_c$ is called a delta-system $\Delta(p, h, c)$ if there is a set F such that $|F| = p$ and $E_i \cap E_j = F$ $\forall i \neq j$. In this paper we find a necessary and sufficient condition for the existence of a decomposition of the complete 3-uniform hypergraph K_n^3 into delta-systems $\Delta(1, 3, c)$. This way we settle a conjecture by Mouyart and Sterboul. Moreover, we solve problems of the maximum (respectively minimum) number of delta-systems $\Delta(1, 3, c)$ to be packed into K_n^3 (resp. to cover K_n^3) for "most" pairs of integers n, c. In the remaining cases we estimate the numbers. © 1990 Academic Press, Inc.

1. INTRODUCTION

By a *hypergraph* we mean a pair $(X, \mathscr{E})$, where X is a finite set (called the set of *vertices*) and $\mathscr{E}$ is a collection of subsets of X (called the set of *edges*). A hypergraph is *h-uniform* if its every edge has h elements. A *graph* is a 2-uniform hypergraph. We say that a hypergraph H is *generated* by edges $E_1, ..., E_c$ if $H = (\bigcup_{i=1}^{c} E_i, \{E_1, ..., E_c\})$.

A *complete* hypergraph K_n^h is an h-uniform hypergraph generated by all h-element subsets of an n-element set. An h-uniform hypergraph generated by edges $E_1, ..., E_c$ is called a *delta-system* $\Delta(p, h, c)$ if there is a set F, called its *center*, such that $|F| = p$ and $E_i \cap E_j = F$ $\forall i \neq j$.

By a *decomposition* of a hypergraph $H = (X, \mathscr{E})$ into hypergraphs $H_1, ..., H_q$ we mean a partition of the set of edges $\mathscr{E}$ into subsets $\mathscr{E}_1, ..., \mathscr{E}_q$ such that H_i is generated by $\mathscr{E}_i$, for $i = 1, ..., q$.

The problem of existence of a decomposition of K_n^h into isomorphic delta-systems $\Delta(p, h, c)$ has been considered by several authors. Baranyai [1] found a necessary and sufficient condition for existence of a decomposition of K_n^h into delta-systems $\Delta(0, h, c)$.

Yamamoto and Tazawa [8] examined the case $p = h - 1$ in connection

0097-3165/90 $3.00

with data base file organization systems. They found some sufficient conditions for a decomposition of K_n^h into delta-systems $\Delta(h-1, h, c)$ to exist. Further results in this direction were obtained by Lonc [3] and Mouyart [4].

The case $h=2$ was solved completely by Yamamoto *et al.* [7], who proved that a complete graph K_n can be decomposed into delta-systems $\Delta(1, 2, c)$ (i.e., into stars of size c) if and only if $c \leqslant n/2$.

Mouyart and Sterboul [5, 6] raised the problem of finding a decomposition of K_n^h into delta-systems $\Delta(p, h, c)$ for arbitrary p, h, and c. The authors found some sufficient conditions for such decomposition to exist. They noted that the general problem is difficult for it contains an existence problem of some Steiner t-designs. The main part of their work is, however, devoted to the case $p=1$ and $h=3$. Among other things they found, for $c \leqslant 6$, a necessary and sufficient condition for existence of a decomposition of K_n^3 into delta-systems $\Delta(1, 3, c)$. In [4] Mouyart conjectures that the conditions are correct for arbitrary c, i.e., the assumption $c \leqslant 6$ can be omitted.

In this paper (see Theorem 1.3) we settle this conjecture.

We shall consider this decomposition problem in a broader packing-covering context. The proof of the conjecture will follow easily from theorems on packing and covering of K_n^3 by delta-systems $\Delta(1, 3, c)$.

A *packing* of hypergraphs $H_1, ..., H_q$ into a hypergraph $H=(X, \mathscr{E})$ is a collection of pairwise disjoint subsets $\mathscr{E}_1, ..., \mathscr{E}_q$ of the edge set $\mathscr{E}$ such that H_i is generated by $\mathscr{E}_i$, for $i=1, ..., q$. A *covering* of a hypergraph $H=(X, \mathscr{E})$ by hypergraphs $H_1, ..., H_q$ is a collection of subsets $\mathscr{E}_1, ..., \mathscr{E}_q$ of the edge set $\mathscr{E}$ such that $\bigcup_{i=1}^q \mathscr{E}_i \supseteq \mathscr{E}$ and H_i is generated by $\mathscr{E}_i$, for $i=1, ..., q$. Clearly, a collection $\mathscr{E}_1, ..., \mathscr{E}_q$ that is both a packing into and a covering of H is a decomposition of H.

Let us denote by $P(n, c)$ (respectively $C(n, c)$) the maximum number (respectively the minimum number) of delta-systems $\Delta(1, 3, c)$ that can be packed into (respectively can cover) K_n^3. It is easy to check that for $n<2c+1$, $P(n, c)=0$ and $C(n, c)$ does not exist. Therefore we shall confine ourselves to the case $n \geqslant 2c+1$.

Our main results are the following three theorems. (Throughout this paper by $\lfloor x \rfloor$ (resp. $\lceil x \rceil$) we mean the greatest integer not greater than x (resp. the least integer not less than x).)

THEOREM 1.1. *If one of the following holds*

(i) $n \geqslant 2c+3$ *and* $c \equiv 0 \pmod 3$,

(ii) $n \geqslant 2c+2$ *and* $c \equiv 1 \pmod 3$,

(iii) $n \geqslant 2c+1$ *and* $c \equiv 2 \pmod 3$, *then*

$$P(n, c) = \left\lfloor \binom{n}{3} \Big/ c \right\rfloor.$$

THEOREM 1.2. *If one of the following holds*

(i) $n \geqslant 2c+2$ *and* $c \equiv 0$ *or* 1 (mod 3)

(ii) $n \geqslant 2c+1$ *and* $c \equiv 2$ (mod 3),

then

$$C(n, c) = \left\lceil \binom{n}{3} \Big/ c \right\rceil.$$

THEOREM 1.3. *The complete hypergraph K_n^3 can be decomposed into delta-systems $\Delta(1, 3, c)$ if and only if*

$$\binom{n}{3} \equiv 0 \qquad (\mathrm{mod}\ c)$$

and one of the following holds

(i) $n \geqslant 2c+1$ *and* $c \not\equiv 1$ (mod 3),

(ii) $n \geqslant 2c+2$ *and* $c \equiv 1$ (mod 3),

(iii) $n = 3$ *and* $c = 1$.

Clearly,

$$P(n, c) \leqslant \left\lfloor \binom{n}{3} \Big/ c \right\rfloor \qquad \text{and} \qquad C(n, c) \geqslant \left\lceil \binom{n}{3} \Big/ c \right\rceil. \tag{1}$$

Thus, to prove Theorems 1.1 and 1.2 it suffices to construct a packing of $\lfloor \binom{n}{3}/c \rfloor$ delta-systems $\Delta(1, 3, c)$ into K_n^3 and a covering of K_n^3 by $\lceil \binom{n}{3}/c \rceil$ delta-systems $\Delta(1, 3, c)$, for n and c satisfying the assumptions of the theorems. Our construction will be recursive. In Section 2 we shall define a hypergraph on n vertices, denoted by $L_{n,c}$, a delta-system decomposition of which implies the existence of appropriate packing into K_{n+1}^3 and covering of K_n^3. Next, we construct recursively a delta-system decomposition of $L_{n+1,c}$, given a delta-system decomposition of $L_{n,c}$.

Section 3 is devoted to finding delta-system decompositions of the hypergraph $L_{n,c}$ for n as small as possible, i.e., close to $2c+1$.

Our three main theorems are derived easily in Section 4 from lemmas and theorems of Sections 2 and 3.

In Theorems 1.1 and 1.2 we establish the numbers $P(n, c)$ and $C(n, c)$ for "most" values of n and c. The final section of this paper is devoted to the cases uncovered by Theorems 1.1 and 1.2. In these cases we usually do not know the exact values of $P(n, c)$ and $C(n, c)$. Nevertheless, we give some bounds for them and we raise conjectures.

Throughout this paper, we denote by $V(H)$, $|H|$, and $e(H)$ the set of vertices, the *order* (i.e., the number of vertices), and the *size* (i.e., the number of edges) of a hypergraph H, resepctively. By $d_H(x)$ we mean the *degree* of x in H, i.e., the number of edges in H containing a vertex x. For a pair of hypergraphs $H_1 = (X_1, \mathscr{E}_1)$, $H_2 = (X_2, \mathscr{E}_2)$, where $\mathscr{E}_2 \subseteq \mathscr{E}_1$, the *difference* $H_1 - H_2$ is the hypergraph generated by the edges $\mathscr{E}_1 - \mathscr{E}_2$. We denote by $G_1 + G_2$ the *disjoint union* of graphs G_1 and G_2. By $\mathscr{P}_k(X)$ we mean the set of all k-element subsets of a set X. All graph-theoretic notions and notation that we use but do not define in this paper can be found in [2].

2. A Recursive Construction

Let us start with a technical lemma.

LEMMA 2.1. *Let G be a graph with the edge chromatic number $\chi'(G)$. If $t \geqslant \chi'(G)$ then G can be decomposed into t matchings of sizes $\lfloor e(G)/t \rfloor$ and $\lceil e(G)/t \rceil$.*

Proof. The proof of this lemma follows easily from a simple observation (see [2, p. 97]): Let M and N be edge-disjoint matchings in G such that $e(M) > e(N)$. Then there are edge-disjoint matchings M' and N' in G such that $e(M') = e(M) - 1$, $e(N') = e(N) + 1$, and $M' \cup N' = M \cup N$.

To prove our lemma consider any decomposition of G into t matchings. It exists because $t \geqslant \chi'(G)$. By repeated application of the above observation we make the sizes of the matchings as equal as possible. The final sizes are $\lfloor e(G)/t \rfloor$ and $\lceil e(G)/t \rceil$. ∎

The hypergraph $L_{n,c}$ defined below will play an important role in our further considerations.

DEFINITION. Let n and c be positive integers such that $n \geqslant 2c+1$ and let $\binom{n}{3} \equiv r \pmod{c}$, where $0 < r \leqslant c$. Denote by $L_{n,c}$ every 3-uniform hypergraph of order $n+1$ with two distinguished vertices x and y such that

(a) the hypergraph generated by all edges of $L_{n,c}$ that do not contain y is isomorphic to K_n^3,

(b) y belongs to $c-r$ edges of $L_{n,c}$,

(c) every edge containing y contains x.

We call the vertex x (respectively y) the *first* (respectively the *second*) *root* of $L_{n,c}$.

It is easy to notice that for fixed n and c, $L_{n,c}$ is unique up to isomorphism. Moreover, $e(L_{n,c}) = \binom{n}{3} + c - r \equiv 0 \pmod{c}$.

The next two lemmas show a close relationship between existence of a decomposition of $L_{n,c}$ into delta-systems $\Delta(1, 3, c)$ and the numbers $P(n+1, c)$ and $C(n, c)$.

LEMMA 2.2. *If $L_{n,c}$ can be decomposed into delta-systems $\Delta(1, 3, c)$ and one of the following holds*

(i) $n \geqslant 2c+2$

(ii) $n \geqslant 2c+1$ *and* $c \not\equiv 0 \pmod 3$,

then

$$P(n+1, c) = \left\lfloor \binom{n+1}{3} \Big/ c \right\rfloor.$$

Proof. In view of (1), it suffices to show the existence of a packing of $\lfloor \binom{n+1}{3}/c \rfloor$ delta-systems $\Delta(1, 3, c)$ into K^3_{n+1}. Clearly, K^3_{n+1} contains a copy of $L_{n,c}$. Let y be the second root of $L_{n,c}$ and denote $H = K^3_{n+1} - L_{n,c}$. Remove y from $V(H)$ and from every edge of H. Denote the resulting graph on n vertices by G. Let us see how many copies of delta-systems $\Delta(1, 3, c)$ can be packed into H. Note that every packing of delta-systems $\Delta(1, 3, c)$ into H is equivalent to packing of matchings of size c into G. Obviously,

$$e(G) = \binom{n+1}{3} - \binom{n}{3} - c + r = \binom{n}{2} - c + r.$$

Let G' be a subgraph of G generated by any $c\lfloor e(G)/c \rfloor$ edges of G. We shall prove, by considering two cases, that

$$\frac{e(G')}{c} \geqslant \chi'(G'). \tag{2}$$

Case 1. $n \geqslant 2c+2$. Note that

$$\frac{e(G')}{c} = \left\lfloor \frac{e(G)}{c} \right\rfloor = \left\lfloor \frac{\binom{n}{2} - c + r}{c} \right\rfloor \geqslant n \geqslant \chi'(G').$$

Case 2. $n = 2c+1$ and $c \not\equiv 0 \pmod 3$. In this case

$$\binom{n}{3} = \frac{(2c+1)\,2c(2c-1)}{6} = c^3 + c\,\frac{c^2-1}{3} \equiv c \qquad \pmod{c},$$

so $r=c$. Consequently,

$$\frac{e(G')}{c}=\left\lfloor\frac{\binom{n}{2}-c+r}{c}\right\rfloor=\left\lfloor\frac{\binom{2c+1}{2}}{c}\right\rfloor=n\geqslant\chi'(G').$$

This completes the proof of (2).

By (2) and Lemma 2.1 G' can be decomposed into $t=\lfloor e(G)/c\rfloor$ matchings of size c. Thus, there is a packing of $\lfloor e(G)/c\rfloor$ delta-systems $\Delta(1,3,c)$ into H.

Since K^3_{n+1} is an edge-disjoint union of $L_{n,c}$ and H and since $L_{n,c}$ can be decomposed into delta-systems $\Delta(1,3,c)$, it is possible to pack

$$\frac{e(L_{n,c})}{c}+\left\lfloor\frac{e(G)}{c}\right\rfloor=\frac{\binom{n}{3}+c-r}{c}+\left\lfloor\frac{\binom{n}{2}-c+r}{c}\right\rfloor$$
$$=\left\lfloor\frac{\binom{n}{3}+c-r+\binom{n}{2}-c+r}{c}\right\rfloor=\left\lfloor\frac{\binom{n+1}{3}}{c}\right\rfloor$$

delta-systems $\Delta(1,3,c)$ into K^3_{n+1}. Therefore, $P(n+1,c)=\lfloor\binom{n+1}{3}/c\rfloor$. ∎

LEMMA 2.3. *If $L_{n,c}$ can be decomposed into delta-systems $\Delta(1,3,c)$ and $n\geqslant 2c+1$ then*

$$C(n,c)=\left\lceil\binom{n}{3}\Big/c\right\rceil.$$

Proof. In view of (1), it suffices to show that K^3_n can be covered by $\lceil\binom{n}{3}/c\rceil$ delta-systems $\Delta(1,3,c)$. Consider a hypergraph $L_{n,c}$ with the first root x and the second root y. Clearly, $L_{n,c}$ contains a copy of K^3_n. Let $H=L_{n,c}-K^3_n$ and denote the edges of H by $xyw_1,\ldots,xyw_{c-r}$. Let $\mathscr{D}$ be a decomposition of $L_{n,c}$ into delta-systems $\Delta(1,3,c)$ and let Δ_j be a member of $\mathscr{D}$ containing xyw_j, for $j=1,\ldots,c-r$. Note that $\Delta_i\neq\Delta_j$, for $i\neq j$, and y is not a center of a delta-system $\Delta(1,3,c)$ generated by any Δ_j.

For $j=1,\ldots,c-r$, delete the edge xyw_j from Δ_j and denote the resulting set by Δ'_j. The collection $(\mathscr{D}-\{\Delta_1,\ldots,\Delta_{c-r}\})\cup\{\Delta'_1,\ldots,\Delta'_{c-r}\}$ is a covering of K^3_n by delta-systems $\Delta(1,3,c)$ and $\Delta(1,3,c-1)$. Since $|\bigcup\Delta'_j|=2c-1\leqslant n-2$, each of the sets Δ'_j can be completed to form a set generating a delta-system $\Delta(1,3,c)$ contained in K^3_n. This way we get a covering of K^3_n by $|\mathscr{D}|=e(L_{n,c})/c=(\binom{n}{3}+c-r)/c=\lceil\binom{n}{3}/c\rceil$ delta-systems $\Delta(1,3,c)$. ∎

The next lemma gives a recursive construction of a decomposition of $L_{n,c}$ into delta-systems $\Delta(1,3,c)$.

LEMMA 2.4. *If $n\geqslant 2c+1$ and $L_{n,c}$ can be decomposed into delta-systems $\Delta(1,3,c)$ then $L_{n+1,c}$ can be decomposed into delta-systems $\Delta(1,3,c)$.*

Proof. Let $\binom{n+1}{3} \equiv r' \pmod c$ and $\binom{n}{3} \equiv r \pmod c$, where $0 < r, r' \leqslant c$. Denote by $v_1, ..., v_n$ the vertices of $L_{n+1,c}$ and assume that $y' = v_{n+2}$ and $x' = v_{n+1}$ are the second and the first roots of $L_{n+1,c}$, respectively. Assume that $x'y'v_1, ..., x'y'v_{c-r'}$ are the edges in $L_{n+1,c}$ containing y'. Clearly, $L_{n+1,c}$ contains a copy of $L_{n,c}$ such that $y = v_{n+1} = x'$ and $x = v_n$ are the second and the first roots of $L_{n,c}$, respectively, and $xyv_1, ..., xyv_{c-r}$ are the edges in $L_{n,c}$ containing y.

To prove the lemma it suffices to show that $L = L_{n+1,c} - L_{n,c}$ can be decomposed into delta-systems $\Delta(1, 3, c)$. Note that every edge of L contains $y = x'$. Let G be a graph on $n+1$ vertices obtained from L by deleting y from $V(L)$ and from every edge of L. Since every decomposition of G into matchings of size c corresponds to a decomposition of L into delta-systems $\Delta(1, 3, c)$, we shall show that G can be decomposed into matchings of size c. To this end observe that

$$e(G) = \binom{n+1}{3} + c - r' - \binom{n}{3} - c + r = \left(\binom{n+1}{3} - r'\right) - \left(\binom{n}{3} - r\right)$$
$$= \binom{n}{2} + r - r' \equiv 0 \pmod c. \tag{3}$$

Considering two cases, we shall prove that

$$\chi'(G) \leqslant \frac{e(G) + c - 1}{c} \tag{4}$$

Case 1. $r > r'$. Note that by (3) and the assumption $n \geqslant 2c+1$ we get

$$\frac{e(G) + c - 1}{c} \geqslant \frac{\binom{n}{2} + 1 + c - 1}{c} \geqslant n + 1.$$

On the other hand, $\chi'(G) \leqslant \Delta(G) + 1 \leqslant n + 1$ so (4) holds in this case.

Case 2. $r \leqslant r'$. In this case $c - r' \leqslant c - r$ so no vertex of G is joined to both x and y'. Thus, $\chi'(G) \leqslant \Delta(G) + 1 \leqslant n$. Moreover, by (3) and the assumption $n \geqslant 2c+1$

$$\frac{e(G) + c - 1}{c} = \frac{\binom{n}{2} + r - r' + c - 1}{c} \geqslant \frac{\binom{n}{2} + 1 - c + c - 1}{c} \geqslant n.$$

This completes the proof of (4).

By (4) and (3)

$$\chi'(G) = \lfloor \chi'(G) \rfloor \leqslant \left\lfloor \frac{e(G) + c - 1}{c} \right\rfloor = \frac{e(G)}{c}.$$

It follows from Lemma 2.1 that G can be decomposed into $e(G)/c$ matchings of size c. This decomposition yields the desired decomposition of L into delta-systems $\Delta(1, 3, c)$. ▮

3. Packing and Covering For n Close to $2c+1$

In view of Lemmas 2.2, 2.3, and 2.4, in order to prove Theorems 1.1, 1.2, and 1.3 we shall need decompositions of $L_{n,c}$ for n as small as possible, i.e., for n close to $2c+1$. Let us start with a construction that will be useful many times in the sequel.

Throughout this section p is a positive integer and $q = \lfloor (p-1)/3 \rfloor$. Consider the complete hypergraph K_p^3 with the set of vertices

$$V = \{1, ..., p\}.$$

Denote

$$A_i = \{i+1, i+2, ..., i+q\},$$
$$B_i = \{i-1, i-2, ..., i-q\},$$

and

$$C_i = V - (A_i \cup B_i \cup \{i\}).$$

Throughout this section all additions and subtractions of the elements of the set V are modulo its cardinality $|V|$.

For $i = 1, ..., p$, denote by F_i a delta-system with center $\{i\}$ generated by the set of edges

$$\mathscr{F}_i = \{ivw \in \mathscr{P}_3(V) \mid v \in A_i \text{ and } w \in B_i\} \cup \{ivw \in \mathscr{P}_3(V) \mid v, w \in C_i\}. \quad (5)$$

Lemma 3.1. (a) *If $p \not\equiv 0 \pmod 3$ then the collection of sets $\mathscr{F}_1, ..., \mathscr{F}_p$ is a decomposition of the complete hypergraph K_p^3.*

(b) *If $p \equiv 0 \pmod 3$ then the collection of sets $\mathscr{F}_1, ..., \mathscr{F}_p$ is a covering of the complete hypergraph K_p^3 such that every edge of K_p^3 is covered exactly once except from the edges belonging to*

$$\mathscr{T} = \{i(i+q+1)(i+2q+2) \mid i = 1, ..., p\}.$$

Every edge in $\mathscr{T}$ is covered exactly three times.

Proof. First we show that in both cases every triple $\alpha\beta\gamma \in \mathscr{P}_3(V)$ is covered at least once.

Assume that $\alpha < \beta < \gamma$ and consider the differences $\beta - \alpha$, $\gamma - \beta$, and $\alpha - \gamma$. Clearly, the differences sum to p. Therefore, at least one of them is greater than q. First, suppose that exactly one of them, say $\beta-\alpha$, is greater than q. In this case $\gamma-\beta$, $\alpha-\gamma \leqslant q$ so $\alpha \in A_\gamma$ and $\beta \in B_\gamma$. Thus, $\alpha\beta\gamma \in \mathscr{F}_\gamma$. Now, assume that at least two of the differences, say $\beta-\alpha$ and $\gamma-\beta$, are greater than q. Note that $\alpha-\beta=p-(\beta-\alpha)=(\gamma-\beta)+(\alpha-\gamma)>q$. Thus, $\alpha \notin A_\beta$ and $\alpha \notin B_\beta$ so $\alpha \in C_\beta$. Similarly, $\gamma \in C_\beta$ because both $\gamma-\beta$ and $\beta-\gamma$ are greater than q. Consequently, $\alpha\beta\gamma \in \mathscr{F}_\beta$. We have shown that $\mathscr{F}_1, ..., \mathscr{F}_p$ is a covering of K_p^3.

To show that $\mathscr{F}_1, ..., \mathscr{F}_p$ is a decomposition of K_p^3 in case (a) note that if $p \not\equiv 0 \pmod 3$ then

$$\sum_{i=1}^{p} |\mathscr{F}_i| = p\left(q^2+\binom{p-2q-1}{2}\right)=\binom{p}{3}=e(K_p^3)$$

so no triple of V occurs more than once in the sets $\mathscr{F}_1, ..., \mathscr{F}_p$.

To show part (b) of the lemma note that for every triple $\alpha\beta\gamma \in \mathscr{T}$ the differences $\beta-\alpha$, $\gamma-\beta$, $\beta-\gamma$ are equal to $q+1$. Thus, $\alpha, \beta \in C_\gamma$, $\beta, \gamma \in C_\alpha$, and $\beta, \alpha \in C_\beta$. Consequently, $\alpha\beta\gamma$ is a member of exactly three sets: $\mathscr{F}_\alpha$, $\mathscr{F}_\beta$, and $\mathscr{F}_\gamma$. Since $p \equiv 0 \pmod 3$,

$$\sum_{i=1}^{p} |\mathscr{F}_i| = p\left(q^2+\binom{p-2q-1}{2}\right)=\binom{p}{3}+\frac{2}{3}p=e(K_p^3)+2\,|\mathscr{T}|.$$

This calculation shows that there are no more repetitions of the edges except from the edges of $\mathscr{T}$ in the covering. ∎

In the next three theorems we use the above lemma to construct decompositions of hypergraphs $L_{n,c}$ into delta-systems $\Delta(1, 3, c)$, where n is close to $2c+1$. The constructions are slightly different according to the divisibility of c by 3.

THEOREM 3.2. *If $c \equiv 2 \pmod 3$ then $L_{2c+1,c}$ can be decomposed into delta-systems $\Delta(1, 3, c)$.*

Proof. Note that $\binom{2c+1}{3}=(2c+1)\,2c(2c-1)/6 \equiv c \pmod c$ so $L_{2c+1,c}$ consists of two components: a copy of K_{2c+1}^3 and a single vertex. Since isolated vertices are nonessential in decompositions, we shall decompose K_{2c+1}^3 into delta-systems $\Delta(1, 3, c)$. Assume that the set of vertices of K_{2c+1}^3 is $V=\{1, 2, ..., 2c+1\}$.

By Lemma 3.1 applied for $p=2c+1$, K_p^3 can be decomposed into hypergraphs $F_1, ..., F_p$ generated by the sets $\mathscr{F}_i$ defined by (5).

To complete the proof it suffices to decompose every hypergraph F_i into delta-systems $\Delta(1, 3, c)$. Let G_i be a graph with the set of vertices $V-\{i\}$ and the set of edges $\{vw \mid vwi \in \mathscr{F}_i\}$. Clearly, G_i is isomorphic to $K_{q,q}+K_{q+1}$, where $q=\lfloor (p-1)/3 \rfloor=(2c-1)/3$. Since q is odd, G_i can be decomposed into q spanning matchings. Every spanning matching in G_i corresponds to a delta-system $\Delta(1, 3, c)$ in F_i. Thus, we can decompose every hypergraph F_i into delta-systems $\Delta(1, 3, c)$. ∎

In the case $c \equiv 1 \pmod 3$ we shall use the following result by Mouyart and Sterboul (see [5, Theorem 2.1, p. 140]).

LEMMA 3.3. *Let $c > 1$. If K^3_{2c+1} can be decomposed into delta-systems $\Delta(1, 3, c)$ then K^3_{2c} can be decomposed into delta-systems $\Delta(1, 3, c-1)$.* ∎

THEOREM 3.4. *If $c \equiv 1 \pmod 3$ then $L_{2c+2,c}$ can be decomposed into delta-systems $\Delta(1, 3, c)$.*

Proof. Note that

$$\binom{2c+2}{3} = \frac{(2c+2)(2c+1)}{3} c \equiv c \qquad (\mathrm{mod}\ c)$$

so $L_{2c+2,c}$ consists of two components: a copy of K^3_{2c+2} and a single vertex. According to Theorem 3.2 K^3_{2c+3} can be decomposed into delta-systems $\Delta(1, 3, c+1)$ so by Lemma 3.3 K^3_{2c+2} can be decomposed into delta-systems $\Delta(1, 3, c)$. ∎

The last case, $c \equiv 0 \pmod 3$, is the most complicated one. Two technical lemmas are necessary.

LEMMA 3.5. *The graph $K_{2k+1} - K_{1,k}$ can be decomposed into $2k$ matchings of size k.*

Proof. Since $\chi'(K_{2k}) = 2k-1$, by Lemma 2.1 the graph K_{2k} can be decomposed into k matchings $M_1, \ldots, M_k$ of size $\lfloor \binom{2k}{2}/2k \rfloor = k-1$ and k matchings $M_{k+1}, \ldots, M_{2k}$ of size $\lceil \binom{2k}{2}/2k \rceil = k$. Clearly, every vertex of K_{2k} belongs to all but one of the matchings M_i, $i = 1, \ldots, k$. Suppose that $v_i \in V(K_{2k}) - V(M_i)$, for $i = 1, \ldots, k$. Adjoin a new vertex v to the set of vertices of K_{2k} and add k edges $vv_1, \ldots, vv_k$. The resulting graph is isomorphic to $K_{2k+1} - K_{1,k}$. The sets of edges of the matchings $M_1 + vv_1$, $M_2 + vv_2, \ldots, M_k + vv_k, M_{k+1}, \ldots, M_{2k}$ form the desired decomposition of $K_{2k+1} - K_{1,k}$. ∎

Let $V = \{1, \ldots, 6k+1\}$, $v \notin V$, and suppose that I is a k-element subset of V. We denote by G_I every graph generated by the set of edges

$$\{wu \in \mathscr{P}_2(V) \mid w, u \in V \text{ and } 0 < w - u \leqslant 2k\} \cup \{vw \mid w \in I\}.$$

LEMMA 3.6. *For some $I \in \mathscr{P}_k(V)$ the graph G_I can be decomposed into $4k+1$ matchings of size $3k$.*

Proof. Note that $G_I - v$ is $4k$-regular. According to the Vizing theorem there is a decomposition of $G_I - v$ into $4k+1$ matchings $M_1, ..., M_{4k+1}$ of sizes at most $3k$. Clearly, for every matching M_i there is a vertex v_i such that $v_i \in V(G_I) - V(M_i)$. The vertices $v_1, ..., v_{4k+1}$ are distinct because the degree of a vertex in $G_I - v$ is $4k$.

Let $I = \{v_1, ..., v_k\}$. The collection of the edge sets of the matchings $M_1 + vv_1$, $M_2 + vv_2, ..., M_k + vv_k, M_{k+1}, ..., M_{4k+1}$ proves that $\chi'(G_I) \leqslant 4k+1$, then apply Lemma 2.1. ∎

THEOREM 3.7. *If* $c \equiv 0 \pmod 3$ *then* $L_{2c+2,c}$ *can be decomposed into delta-systems* $\Delta(1, 3, c)$.

Proof. Let $V = \{1, ..., 2c+1\}$ and suppose that $I \in \mathscr{P}_{c/3}(V)$ satisfies Lemma 3.6, i.e., the graph G_I can be decomposed into $\frac{4}{3}c + 1$ matchings of size c. Let $x, y \notin V$, $x \neq y$.

Since $\binom{2c+2}{3} \equiv \frac{2}{3}c \pmod c$, the hypergraph generated by the set of edges

$$\mathscr{E} = \mathscr{P}_3(V \cup \{x\}) \cup \{vxy \mid v \in I\}$$

is obviously isomorphic to $L_{2c+2,c}$ with the first root x and the second root y.

According to Lemma 3.1, the complete subhypergraph K^3_{2c+1} of $L_{2c+2,c}$ generated by the edges of $\mathscr{P}_3(V)$ can be decomposed into hypergraphs $F_1, ..., F_{2c+1}$ generated by the sets of edges $\mathscr{F}_i$ defined by (5). Clearly, every edge in $\mathscr{X} = \mathscr{E} - \mathscr{P}_3(V)$ contains x. Define

$$\mathscr{D}_i = \{ixv \in \mathscr{P}_3(V \cup \{x\}) \mid \tfrac{2}{3}c < v - i \leqslant c\}$$

for $i = 1, ..., 2c+1$ and denote $\mathscr{D} = \mathscr{X} - \bigcup_{i=1}^{2c+1} \mathscr{D}_i$. Note that $\mathscr{D}, \mathscr{D}_1, \mathscr{D}_2, ..., \mathscr{D}_{2c+1}$ is a decomposition of the hypergraph generated by $\mathscr{X}$. In fact, if $ixv \in \mathscr{D}_i$ then $vxi \notin \mathscr{D}_v$ since $c + 1 \leqslant i - v = 2c + 1 - (v - i) < \frac{4}{3}c + 1$.

Let $\mathscr{H}_i = \mathscr{F}_i \cup \mathscr{D}_i$, for $i = 1, ..., 2c+1$. Note that $\mathscr{H}_1, ..., \mathscr{H}_{2c+1}, \mathscr{D}$ is a decomposition of $L_{2c+2,c}$. Moreover, every edge in $\mathscr{H}_i$ contains i and every edge of $\mathscr{D}$ contains x.

Now, we show that the hypergraph H_i generated by $\mathscr{H}_i$ can be decomposed into delta-systems $\Delta(1, 3, c)$, for $i = 1, ..., 2c+1$. Let G_i be a graph generated by the set of edges $\{vw \mid vwi \in \mathscr{H}_i\}$. It is routine to check that G_i is isomorphic to

$$K_{(2/3)c,\,(2/3)c} + (K_{(2/3)c+1} - K_{1,\,(1/3)c}).$$

Obviously, $K_{(2/3)c,\,(2/3)c}$ can be decomposed into $\frac{2}{3}c$ matchings of size $\frac{2}{3}c$. By Lemma 3.5 $K_{(2/3)c+1} - K_{1,(1/3)c}$ can be decomposed into $\frac{2}{3}c$ matchings of size $\frac{1}{3}c$. These two decompositions combined form a decomposition of G_i into $\frac{2}{3}c$ matchings of size c. This decomposition corresponds to a decomposition of H_i into delta-systems $\Delta(1, 3, c)$.

It remains to be shown that the hypergraph D generated by the edges of $\mathscr{D}$ can be decomposed into delta-systems $\Delta(1, 3, c)$. Again, it is routine to check that the edges $\{vw \mid xvw \in \mathscr{D}\}$ generate the graph G_l. Thus, by Lemma 3.6, it can be decomposed into $\frac{4}{3}c + 1$ matchings of size c. As before this decomposition corresponds to a decomposition of D into delta-systems $\Delta(1, 3, c)$. ▮

4. Proofs of the Main Results

The Theorems 1.1 and 1.2 are now easy conclusions of Lemmas 2.2, 2.3, and 2.4 and Theorems 3.2, 3.4, and 3.7.

Proof of Theorem 1.1. In view of (1), it suffices to show that there exists a packing of $\lfloor \binom{n}{3}/c \rfloor$ delta-systems $\Delta(1, 3, c)$ into K_n^3.

Let us consider the cases (i)–(iii).

(i) The assertion follows immediately by induction from Theorem 3.7 and Lemmas 2.4 and 2.2.

(ii) If $n \geqslant 2c + 3$, we apply Theorem 3.4 and Lemmas 2.4 and 2.2. For $n = 2c + 2$ our theorem follows directly from Theorem 3.4 because $L_{2c+2,c}$ is isomorphic to $K_{2c+2} + K_1$.

(iii) If $n \geqslant 2c + 2$ then we get the assertion by Theorem 3.2 and Lemmas 2.4 and 2.2. For $n = 2c + 1$ the theorem follows directly from Theorem 3.2 because $L_{2c+1,c}$ is isomorphic to $K_{2c+1} + K_1$. ▮

The proof of Theorem 1.2 is very similar to the proof of Theorem 1.1 so we leave it to the reader.

Proof of Theorem 1.3. Obvious necessary conditions for the existence of a decomposition of K_n^3 into delta systems $\Delta(1, 3, c)$ are $\binom{n}{3} \equiv 0 \pmod{c}$ and $n \geqslant 2c + 1$. Suppose that for $c \equiv 1 \pmod{3}$, $c > 1$, there is a decomposition of K_{2c+1}^3 into delta-systems $\Delta(1, 3, c)$. By Lemma 3.3, K_{2c}^3 can be decomposed into delta-systems $\Delta(1, 3, c - 1)$. This is, however, impossible for $\binom{2c}{3} \not\equiv 0 \ (\mathrm{mod}(c - 1))$. The above contradiction completes the proof of necessity.

To show sufficiency, we apply Theorem 1.2. The assumptions of this theorem are satisfied because $\binom{2c+1}{3} \not\equiv 0 \pmod{c}$ for $c \equiv 0 \pmod{3}$. Thus, $C(n, c) = \binom{n}{3}/c$ and we are done since every covering of K_n^3 by $\binom{n}{3}/c$ delta-systems $\Delta(1, 3, c)$ is a decomposition of K_n^3 into delta-systems $\Delta(1, 3, c)$. ▮

5. The Remaining Cases

Observe that all numbers $P(n, c)$ and $C(n, c)$ are determined in Theorems 1.1 and 1.2 except for $P(2c+2, c)$ for $c \equiv 0 \pmod 3$ and $P(2c+1, c)$, $C(2c+1, c)$ for $c \equiv 0$ or $1 \pmod 3$. In this section we shall deal with these remaining cases.

First, let us consider the case of $P(2c+2, c)$, for $c \equiv 0 \pmod 3$. We are not able to find the exact value of $P(2c+2, c)$ in this case. Nevertheless, we establish some bounds.

THEOREM 5.1. *If* $c \equiv 0 \pmod 3$ *then*

$$\tfrac{4}{3}c^2 + \tfrac{5}{3}c + 1 \leqslant P(2c+2, c) \leqslant \tfrac{4}{3}c^2 + 2c.$$

Proof. The upper bound is trivial, just $\lfloor \binom{2c+2}{3}/c \rfloor$. The lower bound follows from Theorem 3.7. By this theorem $L_{2c+2,c}$ can be decomposed into $(\binom{2c+2}{3} + \frac{1}{3}c)/c = \frac{4}{3}c^2 + 2c + 1$ delta-systems $\Delta(1, 3, c)$. The second root y of $L_{2c+2,c}$ belongs to $\frac{1}{3}c$ edges of $L_{2c+2,c}$. Thus, there are at most $\frac{1}{3}c$ delta-systems $\Delta(1, 3, c)$ containing y in the decompostion. The sets of edges of the remaining $\frac{4}{3}c^2 + 2c + 1 - \frac{1}{3}c = \frac{4}{3}c^2 + \frac{5}{3}c + 1$ delta-systems $\Delta(1, 3, c)$ form a packing into K^3_{2c+2}. ∎

Examination of values of $P(2c+2, c)$ for small c inclines us to set the following conjecture.

Conjecture. For $c \equiv 0 \pmod 3$

$$P(2c+2, c) = \tfrac{4}{3}c^2 + 2c.$$

Now, let us consider the numbers $P(2c+1, c)$ and $C(2c+1, c)$. These cases correspond to packing and covering of K^3_n by spanning delta-systems.

The following lemma will be useful in finding a lower bound for $C(2c+1, c)$ and an upper bound for $P(2c+1, c)$.

LEMMA 5.2. (i) *If k delta-systems $\Delta(1, 3, c)$ can be packed into K^3_{2c+1} then*

$$\left\lceil \frac{k}{2c+1} \right\rceil \leqslant 2c+1 - \frac{k-1}{c-1}.$$

(ii) *If K^3_{2c+1} can be covered by k delta-systems $\Delta(1, 3, c)$ then*

$$\left\lfloor \frac{k}{2c+1} \right\rfloor \geqslant 2c+1 - \frac{k-1}{c-1}.$$

Proof. Since the proofs of parts (i) and (ii) are very similar we confine ourselves to the proof of part (i).

Consider the hypergraph H generated by the edges of the k delta-systems $\Delta(1, 3, c)$ packed into K^3_{2c+1}. Let x be a vertex of maximum degree in H. Denote by p the number of these delta-systems $\Delta(1, 3, c)$ in the packing that have their centers in x. Since x belongs to the set of vertices of every delta-system $\Delta(1, 3, c)$ in the packing, we have $d_H(x) = cp + k - p$. Note that

$$\frac{3ck}{2c+1} = \frac{3e(H)}{|H|} = \sum_{v \in V(H)} d_H(v) \Big/ |H| \leqslant d_H(x) = (c-1)p + k$$
$$\leqslant d_{K^3_{2c+1}}(x) = \binom{2c}{2}.$$

It follows from the above inequalities that

$$\frac{k}{2c+1} \leqslant p \leqslant 2c + 1 - \frac{k-1}{c-1}.$$

Since p is an integer

$$\left\lceil \frac{k}{2c+1} \right\rceil \leqslant 2c + 1 - \frac{k-1}{c-1}. \quad \blacksquare$$

The upper bounds for $C(2c+1, c)$ and the lower bounds for $P(2c+1, c)$ will be established by constructions. The next two theorems summarize all results that we have on the numbers $P(2c+1, c)$ and $C(2c+1, c)$.

THEOREM 5.3. *We have*

(i) $\frac{4}{3}c^2 + \frac{2}{3}c - 1 \leqslant C(2c+1, c) \leqslant \frac{4}{3}c^2 + \frac{2}{3}c$ *if* $c \equiv 0 \pmod 3$

(ii) $\frac{4}{3}c^2 + \frac{1}{3}c - \frac{2}{3} \leqslant C(2c+1, c) \leqslant \frac{4}{3}c^2 + \frac{4}{3}c + \frac{1}{3}$ *if* $c \equiv 1 \pmod 3$

(iii) $C(2c+1, c) = \frac{4}{3}c^2 - \frac{1}{3}$ *if* $c \equiv 2 \pmod 3$

(iv) $C(3, 1) = 1$, $C(5, 2) = 5$, $C(7, 3) = 13$, $C(9, 4) = 22$, $C(11, 5) = 33$.

Proof. (i) If K^3_{2c+1} can be covered with k delta-systems $\Delta(1, 3, c)$ then by Lemma 5.2(ii)

$$\left\lfloor \frac{k}{2c+1} \right\rfloor + \frac{k-1}{c-1} \geqslant 2c + 1. \tag{6}$$

Let $k=(2c+1)q+r$, where $0 \leqslant r < 2c+1$. From (6) we get $3cq \geqslant 2c^2 - c - r$. Thus, $q \geqslant \frac{2}{3}c$ or $q = \frac{2}{3}c - 1$ and $r = 2c$. Consequently, $k \geqslant \frac{4}{3}c^2 + \frac{2}{3}c - 1$.

To construct a covering of K^3_{2c+1} by $\frac{4}{3}c^2 + \frac{2}{3}c$ delta-systems $\Delta(1, 3, c)$ we use Lemma 3.1. By this lemma there is a covering of K^3_{2c+1} by hypergraphs F_i, $i = 1, ..., 2c+1$, defined by (5).

We shall cover every hypergraph F_i by delta-systems $\Delta(1, 3, c)$. Let G_i be a graph generated by edges vw, $vwi \in \mathscr{F}_i$. Note that if $c \equiv 0 \pmod 3$ then G_i is isomorphic to $K_{(2/3)c, (2/3)c} + K_{(2/3)c}$. It is routine to show that this graph can be covered by $\frac{2}{3}c$ matchings of size c. Covering of G_i by matchings of size c corresponds to covering of F_i by delta-systems $\Delta(1, 3, c)$. The total number of delta-systems $\Delta(1, 3, c)$ in the covering of K^3_{2c+1} is $(2c+1)\frac{2}{3}c = \frac{4}{3}c^2 + \frac{2}{3}c$.

(ii) The proof of the lower bound is very similar to the proof in case (i) so we omit it. The proof of the upper bound is analogous too. The graph G_i is, however, isomorphic to $K_{(2/3)(c-1), (2/3)(c-1)} + K_{(2/3)(c-1)+2}$ this time. Again, it is easy to check that this graph can be covered by $\frac{2}{3}(c-1)+1$ matchings of size c. The total number of delta-systems $\Delta(1, 3, c)$ in the covering, defined as in case (i), is $(2c+1)(\frac{2}{3}(c-1)+1) = \frac{4}{3}c^2 + \frac{4}{3}c + \frac{1}{3}$.

(iii) is a special case of Theorem 3.2.

(iv) In view of (iii), the proof is required for $C(7, 3)$ and $C(9, 4)$ only. Since by (i) and (ii) $C(7, 3) \geqslant 13$ and $C(9, 4) \geqslant 22$, it suffices to construct appropriate coverings. The coverings are shown beneath. (The sets of vertices of K^3_7 and K^3_9 are first successive positive integers and every column generates a delta-system $\Delta(1, 3, c)$.)

Covering of K^3_7 by 13 delta-systems $\Delta(1, 3, 3)$.

216	315	413	435	514	547	614	645	712	724	124	125	123
234	327	425	426	573	532	653	632	743	736	136	137	147
257	346	467	417	526	561	627	671	756	715	157	146	156

Covering of K^3_9 by 22 delta-systems $\Delta(1, 3, 4)$.

125	128	213	216	315	317	412	419	514	518	689	683	679
167	193	289	275	396	384	497	472	569	573	674	692	681
134	147	245	249	324	326	435	436	523	529	625	675	624
189	165	267	283	387	395	468	458	578	564	613	614	635

768	763	769	867	879	862	978	964	968
795	782	784	893	865	871	961	973	972
712	791	715	814	813	894	923	985	915
734	745	723	825	824	835	945	912	934 ∎

THEOREM 5.4. *We have*

(i) $\frac{4}{3}c^2 - \frac{4}{3}c - 1 \leqslant P(2c+1, c) \leqslant \frac{4}{3}c^2 - \frac{1}{3}c$ *if* $c \equiv 0 \pmod 3$

(ii) $\frac{4}{3}c^2 - \frac{2}{3}c - \frac{2}{3} \leqslant P(2c+1, c) \leqslant \frac{4}{3}c^2 - \frac{2}{3}c + \frac{1}{3}$ *if* $c \equiv 1 \pmod 3$

(iii) $P(2c+1, c) = \frac{4}{3}c^2 - \frac{1}{3}$ *if* $c \equiv 2 \pmod 3$

(iv) $P(3, 1) = 1$, $P(5, 2) = 5$, $P(7, 3) = 10$, $P(9, 4) = 19$, $P(11, 5) = 33$.

This theorem can be proved using the same methods as those used in the previous theorem. Thus, we leave the proof to the reader.

Based on parts (iv) of Theorems 5.3 and 5.4, we suspect that the lower bounds are exact values of $C(2c+1, c)$ and the upper bounds (except for $c = 3$) are exact values of $P(2c+1, c)$. More precisely, we suppose that the following conjecture is true for $c \neq 3$.

Conjecture. We have

$$\text{(i)}\quad C(2c+1, c) = \begin{cases} \frac{4}{3}c^2 + \frac{2}{3}c - 1 & \text{for } c \equiv 0 \pmod 3 \\ \frac{4}{3}c^2 + \frac{1}{3}c - \frac{2}{3} & \text{for } c \equiv 1 \pmod 3 \\ \frac{4}{3}c^2 - \frac{1}{3} & \text{for } c \equiv 2 \pmod 3. \end{cases}$$

$$\text{(ii)}\quad P(2c+1, c) = \begin{cases} \frac{4}{3}c^2 - \frac{1}{3}c & \text{for } c \equiv 0 \pmod 3 \\ \frac{4}{3}c^2 - \frac{2}{3}c + \frac{1}{3} & \text{for } c \equiv 1 \pmod 3 \\ \frac{4}{3}c^2 - \frac{1}{3} & \text{for } c \equiv 2 \pmod 3. \end{cases}$$

REFERENCES

1. Z. BARANYAI, On the factorization of the complete uniform hypergraph, *in* "Finite and Infinite Sets," pp. 91–108, Colloquia Mathematica Societatis Jánas Bolyai, Vol. 10, North Holland, Amsterdam, 1975.
2. J. A. BONDY AND U. S. R. MURTY, "Graph Theory with Applications," Macmillan & Co., London, 1976.
3. Z. LONC, Decompositions of hypergraphs into hyperstars, *Discrete Math.* **66** (1987), 157–168.
4. A. F. MOUYART, Decomposition of the complete hypergraph into hyperclaws, *Discrete Math.* **76** (1989), 137–150.
5. A. F. MOUYART AND F. STERBOUL, Decomposition of the complete hypergraph into delta-systems, I, *J. Combin. Theory Ser. A* **40** (1985), 290–304.
6. A. F. MOUYART AND F. STERBOUL, Decomposition of the Complete Hypergraph into Delta-Systems, II, *J. Combin. Theory Ser. A* **41** (1986), 139–149.
7. S. YAMAMOTO, H. IKEDA, S. SHIGE-EDA, K. USHIO, AND N. HAMADA, On claw-decomposition of complete graphs and complete bigraphs, *Hiroshima Math. J.* **5** (1975), 33–42.
8. S. YAMAMOTO AND S. TAZAWA, Hyperclaw decomposition of complete hypergraphs, *Ann. Discrete Math.* **6** (1980), 385–391.